Groups of loops and hoops

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Abstract

The approaches to quantum field theories based in the so called loop representation deserved much attention recently. In it, closed curves and holonomies around them play a central role. In this framework the group of loops and the group of hoops have been defined, the first one consisting in closed curves quotient with the equivalence relation that identifies curves differing in retraced segments, and the second one consisting in closed curves quotient with the equivalence relation that identifies curves having the same holonomy for every connection in a fiber bundle. The purpose of this paper is to clarify the relation between hoops and loops, or in other words, to give a description of the class of holonomy equivalent curves.

1 Introduction

An important step in the construction of quantum field theories is the definition of the space of states $L^2(\mathcal{A}/\mathcal{G})$. This is done in [1, 2] by first constructing generalized measures on $\mathcal{A}/\mathcal{G}$. In these constructions the notions of group of loops, holonomy around loop and group of hoops play a central role (the precise definitions are stated below).

Given a differentiable manifold M and a point o of M we construct the space of closed curves in M, Ω, as the set of piecewise regular curves $\alpha : [0,1] \to M$. A curve $\beta : [a,b] \to M$ is regular if there exists $\epsilon > 0$ and a differentiable (or analytic) curve $\gamma : (a-\epsilon, b+\epsilon) \to M$ such that β and γ coincide in $[a,b]$, and we say that a curve $\alpha : [0,1] \to M$ is piecewise regular if exists a partition of $[0,1]$, $0 = t_0 < t_1 < \cdots < t_n = 1$ such that α restricted to each of the intervals $[t_{i-1}, t_i]$ is regular.

In Ω we can define the inverse of a curve $\alpha^{-1}(t) = \alpha(1-t)$, and the composition of curves

$$\alpha\beta(t) = \begin{cases} \alpha(2t) & \text{if } t < 1/2 \\ \beta(2t-1) & \text{if } t \geq 1/2 \end{cases}$$

if we identify curves that only differ in a reparameterization the composition is an associative operation, but in general $\alpha\alpha^{-1} \neq c$ (c being the constant curve), to make Ω in a group we need to introduce a further equivalence relation. One possibility is to identify curves differing in retraced segments, that is we identify $\alpha\beta$ with $\alpha\rho\rho^{-1}\beta$, the group obtained is called the group of loops and is denoted by $\mathcal{LG}$ or $\mathcal{LG}^{\omega}$ if we work with analytic curves. Other possible identification is, given a principal bundle (E, M, G, π), G a Lie group, identify two curves α and β if they have the same holonomy for every connection in the bundle, the group obtained this way is called the group of hoops and is denoted by $\mathcal{HG}$ or $\mathcal{HG}^{\omega}$ in the analytic case. The purpose of this paper is to clarify the relation between $\mathcal{LG}$ and $\mathcal{HG}$ and how $\mathcal{HG}$ depends on the Lie group G. In particular we obtain results for piecewise differentiable loops without making any assumptions on the Lie group.

Consider the infinite set of symbols $e_1, e_2, \ldots$ and $e_1^{-1}, e_2^{-1}, \ldots$, and let E be the set of words in that symbols including the null word, a word is a finite ordered list of symbols (ex. $e_3e_1^{-1}e_2$ is a word). If we define the product of word as the concatenation and identify words that differ by "canceling opposite symbols", that is $w_1e_ie_i^{-1}w_2 \sim w_1w_2$, then E is a free group. Let us define E_G, the group of *identities* of G, as the subgroup of E consisting in words, say $e_2e_3e_1^{-1}$, such that if we assign to each e_i a element g_i of G and multiply these in the way specified by the word (as $g_2g_3g_1^{-1}$) the result is the identity of G no matter what choice of g_i (ex. if G is abelian $e_1e_2e_1^{-1}e_2^{-1}$ is an identity). In other words, E_G is the intersection of the kernels of every homomorphism of groups from E to G

$$E_G = \bigcap_{f \in \hom(E,G)} \ker f.$$

Now we define $E_G(\mathcal{LG})$ as the subgroup of $\mathcal{LG}$ generated by the loops obtained in the following way: for every word in E_G (such as $e_2e_1e_3^{-1}$) and every assignment of a loop α_i to each of the symbols e_i take the product of α_i in the same way as the word (ex. $\alpha_2\alpha_1\alpha_3^{-1}$). Or equivalently

$$E_G(\mathcal{LG}) = \bigcup_{f \in \hom(E,\mathcal{LG})} f(E_G).$$

Now we can state the main results, first in the analytic case.

Theorem 1.1 *For G a connected Lie group, $\mathcal{HG}^{\omega} = \mathcal{LG}^{\omega}/E_G(\mathcal{LG}^{\omega})$.*

This result is complemented with results about E_G of section 3.

Theorem 1.2 *If G is abelian then E_G is generated by elements of the form $e_ie_je_i^{-1}e_j^{-1}$.*

Theorem 1.3 *If G is connected and non solvable then has no non trivial identities.*

Then we have the following corollaries (see [1]).

Corollary 1.4 *If G is abelian then $\mathcal{HG}^\omega = \mathcal{LG}^\omega/[\mathcal{LG}^\omega, \mathcal{LG}^\omega]$.*

Corollary 1.5 *If G is connected and non solvable then $\mathcal{HG}^\omega = \mathcal{LG}^\omega$.*

In section 5 we show through a example that theorem 1.1 is not valid in the differentiable case. However we have:

Theorem 1.6 *For G a connected Lie group, $\mathcal{HG} = \mathcal{LG}/\overline{E_G(\mathcal{LG})}$.*

Where $\overline{E_G(\mathcal{LG})}$ is the closure of $E_G(\mathcal{LG})$ in the quotient topology arising from the C^N topology of curves for any N. The topology of the loop space is discussed in section 4 where we also show that the topology introduced by Barret [3] coincides with the usual C^N topology. However we can generalize corollary 1.5 for a non solvable group even in the case of piecewise differentiable loops, as the following theorem shows.

Theorem 1.7 *If G is connected and non solvable then $\mathcal{HG} = \mathcal{LG}$.*

2 Analytic loops

First we consider the case of analytic loops, this case is simpler because of the way in which analytic curves intersect, they either intersect in finitely many points or they intersect in a segment. From this we obtain a decomposition of a loop in independent loops.

Let us define what we mean by *independent loops*, we say that a loop α has a segment ρ that is traced once if there exists curves β and γ such that $\alpha = \beta\rho\gamma$ and β and γ don't intersect ρ except at the endpoints. A set of loops $\alpha_1, \ldots, \alpha_n$ is independent if each loop α_i has a segment ρ_i traced once and the segments ρ_i do not intersect.

Theorem 2.1 *Every loop can be decomposed in product of independent loops.*

Proof: The loop γ is piecewise analytic thus it can be written as a product of analytic curves $\gamma = \rho_1 \ldots \rho_n$, the curves ρ_i intersect each other in finitely many points or in a common segment thus each ρ_i can be decomposed in segments that intersect only at the endpoints or coincide, then $\gamma = \alpha_{i_1}^{s_1} \ldots \alpha_{i_k}^{s_k}$, where s_j is either 1 or -1.

Let us denote by $e_-(\alpha_i)$ the initial point of α_i and $e_+(\alpha_i)$ the final point. Let E denote the set of all the endpoints of all α_i, for each point $p \in E$ chose a curve $\beta(p)$ from o to p that does not intersect the segments α_i, and let $\gamma_i = \beta(e_-(\alpha_i))\alpha_i\beta(e_+(\alpha_i))$ then $\gamma = \gamma_{i_1}^{e_1} \ldots \gamma_{i_k}^{e_k}$. QED

Lemma 2.2 *Let (E, M, G, π) be a principal bundle with G a connected Lie group and α a loop in M with a segment traced once, chose any element g of G, then there is a connection θ in E such that $H_\theta(\alpha) = g$.*

Proof: The loop α has a segment traced once, thus we can find a local parameterization of M such that

1. its domain contains $I = [0, 1]^n$, in what follows we identify points in I with its images in M and we fix a trivialization of the bundle over I.
2. $\alpha = \beta\gamma\xi$ such that β and ξ don't have points in I except its endpoints.
3. γ is the segment from $a = (0, 1/2, \ldots, 1/2)$ to $b = (1, 1/2, \ldots, 1/2)$ in I.

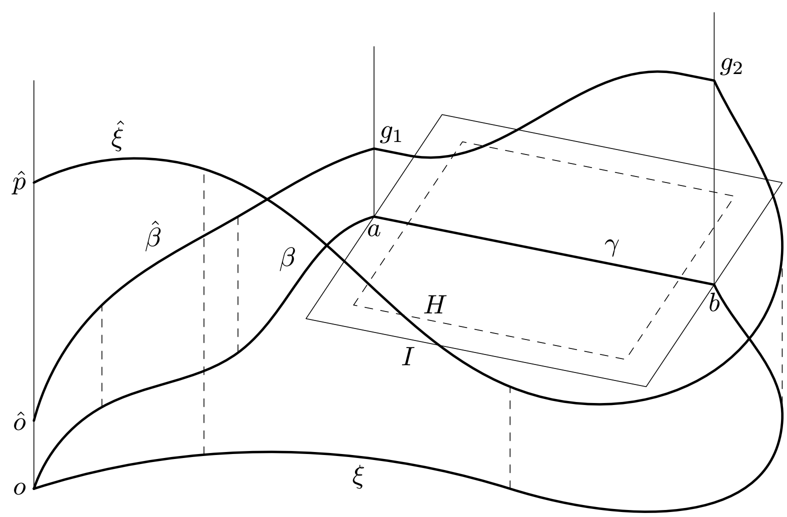

Figure 1:

Let A be any connection, take a small deformation of A such that A is flat over I. Take $\hat{o}$ and $\hat{p}$ in the fiber over o such that $\hat{o} = \hat{p}g$, let $\hat{\beta}$ the horizontal lift of β that starts in $\hat{o}$ and $\hat{\xi}$ the horizontal lift of ξ that ends in $\hat{p}$. Next fix a trivialization of the bundle over I, thus elements in fibers over points of I can be identified with elements of G. Let g_1 be the endpoint of $\hat{\beta}$ in the fiber over a and g_2 the endpoint of $\hat{\xi}$ in the fiber over b, see figure, G is connected then there exists a curve $s : [0, 1] \to G$ such that $s(t) = g_1$ for $t \in [0, \epsilon)$ and $s(t) = g_2$ for t in $(1 - \epsilon, 1]$ and let $\rho : [0, 1] \to \mathbb{R}$ differentiable such that $\rho(t) = 1$ for $t \in [\epsilon, 1 - \epsilon]$ and $g(t) = 0$ for $t \in [0, \epsilon/2) \cup (1 - \epsilon/2, 1]$, take the connection $A'_x = L_{s(x_1)^{-1}} \dot{s}(x_1)\rho(x_2) \ldots \rho(x_n)dx_1$, let B the connection defined as A outside I and as A' in I (note that this is a smooth connection because A is flat in I and the way in which s and ρ were chosen), then $H_B(\alpha) = g$. QED

Note that the proof of this lemma requires to change a given connection only in a small neighborhood of a point in the segment traced once, then we can use it to prove the next proposition.

Proposition 2.3 *Let (E, M, G, Π) be a principal bundle with G a connected Lie group and $\alpha_1, \ldots, \alpha_n$ be independent loops then for every $(g_1, \ldots, g_n)$ in G^n there is a connection A such that $H_A(\alpha_i) = g_i$ for $i = 1, \ldots, n$.*

Now we state and prove the main theorem of the section.

Theorem 2.4 *For G a connected Lie group, $\mathcal{HG}^\omega = \mathcal{LG}^\omega / E_G(\mathcal{LG}^\omega)$.*

Proof: Let α be a loop in $E_G(\mathcal{LG}^\omega)$ then there exist a word in E_G, for example $e_1 e_2 e_1^{-1} e_3 e_2^{-1}$, such that $\alpha = \alpha_1 \alpha_2 \alpha_1^{-1} \alpha_3 \alpha_2^{-1}$ then for every connection A, if we define $g_i = H_A(\alpha_i)$, the holonomy of α is $H_A(\alpha) = g_1 g_2 g_1^{-1} g_3 g_2^{-1} = e$. Conversely if α is a loop not in $E_G(\mathcal{LG}^\omega)$ then by theorem 2.1 there exists independent loops $\alpha_1, \ldots, \alpha_n$ such that $\alpha = \alpha_{i_1}^{s_1} \ldots \alpha_{i_k}^{s_k}$. Note that $e_{i_1}^{s_1} \ldots e_{i_k}^{s_k}$ is not an identity of G because $\alpha \notin E_G(\mathcal{LG}^\omega)$, then there exist $g_1, \ldots, g_n$ in G such that $g_{i_1}^{s_1} \ldots g_{i_k}^{s_k} \neq e$; by proposition 2.3 there exists a connection A such that $H_A(\alpha_i) = g_i$ then $H_A(\alpha) \neq e$. QED

3 Identities in Lie groups

In this section we prove the following theorems

Theorem 3.1 *If G is abelian then E_G is generated by elements of the form $e_i e_j e_i^{-1} e_j^{-1}$.*

Theorem 3.2 *If G is connected and non solvable then it has no non trivial identities.*

Proof of 3.1: If g is abelian then E_G contains all words of the form $e_i e_j e_i^{-1} e_j^{-1}$. Conversely, if $e_{i_1}^{s_1} \ldots e_{i_k}^{s_k}$ is an identity all the words formed by reordering terms are also identities because G is abelian, thus is sufficient to prove that $e_1^{a_1} \ldots e_n^{a_n}$ is an identity iff $a_i = 0$ for $i = 1, \ldots, n$. If $a_j = m \neq 0$ then take g an element of G such that $g^m \neq e$ (for example if v is a vector in $\mathfrak{g}$ such that $\exp v \neq e$ the take $g = \exp v/m$), define $g_i = e$ if $i \neq j$ and $g_j = g$ then $g_1^{a_1} \ldots g_n^{a_n} = g^m \neq e$ thus $e_1^{a_1} \ldots e_n^{a_n}$ is not an identity of G. QED

To prove 3.2 we show that if G is non solvable then for every n it has a free subgroup with n generators. We will use the following theorem due to Tits [7].

Theorem 3.3 *Let $G \subset GL(V)$ be a subgroup, V a finite-dimensional vector space over a field of characteristic 0. Then G has a free subgroup with n generators for every n or G has a solvable subgroup of finite index.*

First we recall the definition of solvable groups. Let G be a group. The derived group G' is the subgroup of G generated by elements of the form $xyx^{-1}y^{-1}$, $x, y \in G$, then define by induction

$$G^{(1)} = G' \qquad G^{(n+1)} = \left(G^{(n)}\right)',$$

then G is *solvable* if $G^{(n)}$ is the trivial group for some n. Next we prove the following theorem.

Theorem 3.4 *If G is a subgroup of $GL(n, \mathbb{R})$ connected and non solvable then for every n it has a free subgroup with n generators.*

Proof: Suppose that G does not contain a free subgroup with n generators, then by theorem 3.3 G has a solvable subgroup of finite index H. Let $\overline{H}$ be the closure of H in G; then $\overline{H}$ is a solvable subgroup of finite index of G, then either $\overline{H} = G$ which is absurd because G is nonsolvable, or the index of $\overline{H}$ in G is greater than 1 then G is union of finitely many closed disjoint subset (the cosets of $\overline{H}$) which is absurd because G is connected. QED

To prove theorem 3.2 we use the adjoint representation of a Lie group, $Ad : G \to \mathrm{Aut}(\mathfrak{g})$, $Ad(g)v = da_g v$ where $a_g : G \to G$, $a_g(x) = gxg^{-1}$.

Proof of 3.2: We need to show that $Ad(G)$ is a connected nonsolvable subgroup of $\mathrm{Aut}(\mathfrak{g})$. Clearly $Ad(G)$ is connected because Ad is continuous. Suppose that $Ad(G)$ is solvable, that is, there exist n such that $Ad(G)^{(n)} = \{e\}$, but $Ad(G)^{(n)} = Ad(G^{(n)})$ then $G^{(n)} \subset \mathrm{Ker}\, Ad = Z(G)$ ($Z(G)$ is the set of all elements in G that commute with every other element of G), then $G^{(n+1)} = \{e\}$ which is absurd because G is nonsolvable. Then $Ad(G)$ is a connected nonsolvable subgroup of $\mathrm{Aut}(\mathfrak{g})$ and by theorem 3.4, $Ad(G)$ has no non trivial identities thus G has no non trivial identities. QED

4 Topology of the loop space

In this section we discuss several ways to give a topology to the loop space, we work in the space of parameterized paths in M, let

$$P^N = \{\gamma : [0,1] \to M : \gamma \text{ is piecewise } C^N\}$$

and we define $P^\infty = \bigcap_{N>0} P^N$. We define the C^N topology in P^N giving a subbase of open sets. Let $\phi : U \subset M \to \mathbb{R}^d$ be a coordinate system in M, $a < b \in [0,1]$ and γ a curve such that $\gamma|_{[a,b]} \subset U$, then we define

$$U^N_{\phi,a,b}(\gamma, \epsilon) = \{\alpha \in P^N : \alpha|_{[a,b]} \subset U \text{ and } |\alpha^{(n)}(x) - \gamma^{(n)}(x)| < \epsilon, \\ \forall n \leq N, x \in [a,b]\}$$

where we identify α with $\phi \circ \alpha$ and we denote by $\alpha^{(n)}$ the n-th derivate of α. We take this family of sets with N fixed as a subbase of the C^N topology in

P^N and take the family of this sets for all N as a subbase of the C^∞ topology in P^∞.

We now give another characterization of this topology. Define a C^N homotopy, where possibly $N=\infty$, as function $\Phi : U \to P^N$ that is obtained from a C^N function $\phi : U \times [0,1] \to M$ with U a open set of $\mathbb{R}^n$, the finest topology in which all the C^N homotopies are continuous is the Barret [3] topology. We affirm that the Barret topology coincides with the C^N topology. We give a proof for the C^∞ case, the C^N case is similar.

In what follows we consider that all the curves are contained in the domain of a coordinate system (if not we can divide paths in smaller pieces) and thus we identify them with paths in $\mathbb{R}^d$. As obviously all the C^∞ homotopies are continuous in the C^∞ topology, closed sets in the C^∞ topology are closed in the topology generated by C^∞ homotopies, the converse follows from the following lemma.

Lemma 4.1 *If γ_n is a sequence of C^∞ curves in $\mathbb{R}^n$ converging in the C^∞ topology to γ then there is a homotopy $\Phi : (-1,1) \to P^\infty$ such that $\Phi(0) = \gamma$ and $\alpha_n = \Phi(2^{-n})$ is a subsequence of γ_n.*

Proof: Take $\rho : [0,1] \to [0,1]$ a C^∞ function such that $\rho(0) = 0$, $\rho(1) = 1$, $\rho^{(n)}(0) = \rho^{(n)}(1) = 0$ for all $n \geq 0$ and let $a_n = \max_{x\in[0,1],k\leq n} |\rho^{(k)}(x)|$. Take α_n a subsequence of γ_n such that $\alpha_n \in U^N(\gamma, 2^{-N^2-N-1}a_N^{-1})$ for all $n \geq N-1$. Define $\phi : (-1,1) \times [0,1] \to \mathbb{R}^d$ as

$$\phi(s,t) = \begin{cases} \gamma(t) & \text{if } s \leq 0 \\ (1-\rho(2^n s - 1))\alpha_n(t) + \rho(2^n s - 1)\alpha_{n-1}(t) & \text{if } 2^{-n} < s \leq 2^{-n+1} \end{cases}$$

Then obviously $\phi(s,t)$ is C^∞ when $s \neq 0$, we have to show that

$$\frac{\partial^{k+l}\phi}{\partial s^k \partial t^l} \to 0$$

when $s \to 0$, take $n > k+l$, then for $2^{-n} < s \leq 2^{-n+1}$ we have

$$\begin{aligned} \left|\frac{\partial^{k+l}\phi}{\partial s^k \partial t^l}\right| &= |-2^{nk}\rho^{(k)}(2^n s - 1)\alpha_n^{(l)}(t) + 2^{nk}\rho^{(k)}(2^n s - 1)\alpha_{n-1}^{(l)}(t)| \\ &\leq 2^{n^2} a_n |\alpha_n^{(l)}(t) - \alpha_{n-1}^{(l)}(t)| \leq 2^{n^2} a_n 2^{-n^2-n} a_n^{-1} \leq 2^{-n}. \end{aligned}$$

Then $\phi(s,t)$ is a C^∞ function thus Φ is a C^∞ holonomy such that $\alpha_n = \Phi(2^{-n})$ and $\gamma = \Phi(0)$. QED

5 Differentiable loops

The key of the proof of theorem 1.1 for analytic loops was the theorem of decomposition of a loop in product of independent loops (theorem 2.1), in the

case of differentiable loops this theorem is not valid because differentiable curves can intersect in complicated ways.

For example let $\rho : [0,1] \to [0,1]$ a differentiable function such that $\rho^{(n)}(0) = \rho^{(n)}(1) = 0$ for all $n \geq 0$ and let $a_n = \max_{x\in[0,1],n>0} |\rho^{(n)}(x)|$. Define $f_1, f_2 : [0,1] \to \mathbb{R}$ as

$$f_i(x) = (3-2i)^n \frac{1}{2^n a_n} \rho(2^n x - 1) \text{ if } 2^{-n} < x \leq 2^{-n+1}, \quad i = 1,2$$

and $f_3 = -f_1$, $f_4 = -f_2$, see figure 2.

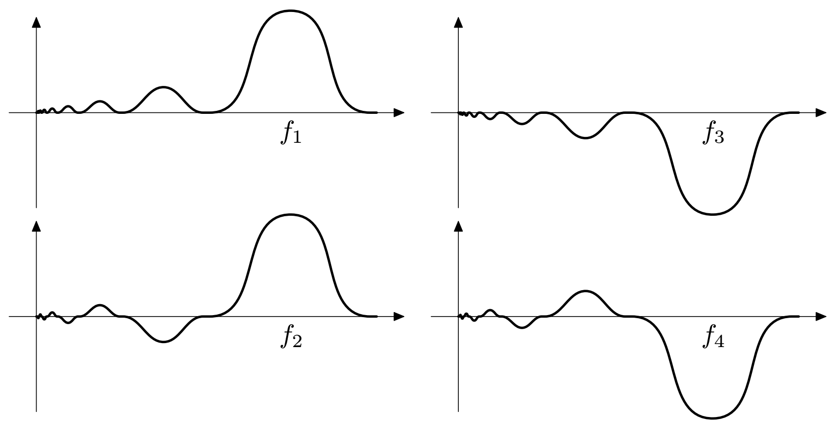

Figure 2:

Take the curves $c_i(x) = (x, f_i(x))$ and $c = c_1 c_2^{-1} c_3 c_4^{-1}$ then c has trivial holonomy for any connection in a bundle with abelian structure group G but it is not in $E_G(\mathcal{LG})$.

However we have another decomposition available [2].

Let us begin with some definitions, let T be a family of curves $\alpha_1, \ldots, \alpha_n$ in M, $\alpha_i : [0,1] \to M$, we define range(T) as the union of the images of the curves in M, a point p of range(T) is a *regular point* if there is a neighborhood U of p such that $U \cap \text{range}(T)$ is an embedded segment in M.

Definition 5.1 *Let p be a regular point, the* type *of p is the subgroup of G^n generated by the elements in which the i-th component is e if α_i do not pass through p and the other components are all equal.*

Definition 5.2 *We say that the family T is a* tassel *based at b if*

1. range(T) *is contained in a contractible open subset of M.*
2. $\alpha_i(0) = b$ *for $i = 1, \ldots, n$.*
3. *there is a parameterization $(x_1, \ldots, x_d) \mapsto M$ such that $b = (0, \ldots, 0)$ and α_i can be written as a graph $\alpha_i(t) = (t, f_i(t))$, where $f_i : [0, t_i] \to \mathbb{R}^{d-1}$.*

4. *if there is a regular point in* range(T) *with a certain type, then there are points with the same type in every neighborhood of* b.

5. *all the curves in the family are different.*

Definition 5.3 *A family of curves* $\alpha_{1,1}, \ldots, \alpha_{1,n_1}, \ldots, \alpha_{k,1}, \ldots, \alpha_{k,n_k}$ *is a* web *if the curves* $\alpha_{j,1} \ldots \alpha_{j,n_j}$ *form a tassel* T_j *for* $j = 1, \ldots, k$ *and curves in two different tassels do not intersect (except, possibly, at their endpoints).*

In [2] is proven the following proposition.

Proposition 5.4 *For every family of curves* F *in* M *there is a web* W *such that all curves in* F *are products of curves in* W *(or their inverses).*

Applying this proposition to a loop γ we obtain a web W formed by a family of curves $\alpha_1, \ldots, \alpha_n$ such that γ can be obtained as product of curves in W, then repeating the same construction as in proof of theorem 2.1 we can construct loops $\gamma_i = \beta(e_-(\alpha_i))\alpha_i\beta(e_+(\alpha_i))$ then γ is a product of loops γ_i.

Definition 5.5 *Let* T *be a tassel,* G_T *is the closed subgroup of* G^n *generated by all the types of regular points in* T.

Now let us see what are the possible holonomies for curves in a tassel, let T be a tassel composed by curves $\alpha_1, \ldots, \alpha_n$, range$(T)$ is contained in a contractible open set U, then we can fix a trivialization of the bundle over U and associate to each connection a element of G over each curve, then we can identify the set of possible values of holonomies for the curves α_i with a subset of G^n, in [2] is also proved:

Proposition 5.6 *For a tassel* T *the set of possible values for the holonomies is* G_T

Lemma 5.7 *If* G *is semisimple then* $G_T = G^n$.

Proof: It is sufficient to show that all the elements of G^n of the form $E(g,i) = (e, \ldots, g, \ldots, e)$ (the element of G^n that has g in the i-th component and e in the others) are in G_T, now we say that a element of G^n is of the form $E(g, i, i_1, \ldots, i_k)$ if its i-th component is g and the components $i_1, \ldots, i_k$ are e. We will show that G_T has elements of the form $E(g, i, i_1, \ldots, i_k)$ for every g, i, $i_1, \ldots, i_k$ (when $k = n-1$ this implies that G_T has all the elements of the form $E(g,i)$ then $G_T = G^n$), we proceed by induction in k, when $k = 1$, given g, i, i_1, we can find a regular point of T such that α_i passes trough p and α_{i_1} does not, then the type of p is a element of G_T of the form $E(g, i, i_1)$. To proceed with the inductive step assume that we are given $g = g_1 g_2 g_1^{-1} g_2^{-1}$, i, $i_1, \ldots, i_{k+1}$. Then take elements of G_T $\hat{g}_1$ of the form $E(g_1, i, i_1, \ldots, i_k)$ and $\hat{g}_2$ of the form $E(g_2, i, i_2, \ldots, i_{k+1})$ (they exist by induction hypothesis), then $\hat{g}_1\hat{g}_2\hat{g}_1^{-1}\hat{g}_2^{-1}$ is a

element of G_T of the form $E(g, i, i_1, \dots, i_{k+1})$. Since G is semisimple elements of the form $g_1 g_2 g_1^{-1} g_2^{-1}$ generate G, then for every $g \in G$ G_T has elements of the form $E(g, i, i_1, \dots, i_{k+1})$. QED

Now we can give a proof of theorem 1.7

Proof of 1.7: Let (E, G, M, Π) be a principal bundle with G a non solvable group, let $\hat{G}$ be the quotient of G by its radical, then $\hat{G}$ is semisimple, let $(\hat{E}, \hat{G}, M, \hat{\Pi})$ be the extension of the bundle to the group $\hat{G}$. Let γ be a loop in M not null in $\mathcal{LG}$ then as remarked before we can decompose γ in product of loops $\gamma_i = \beta(e_-(\alpha_i))\alpha_i\beta(e_+(\alpha_i)$ where the curves $\alpha_1, \dots, \alpha_n$ form a web, then because $\hat{G}$ is semisimple we can choose holonomies independently for the loops γ_i then we can proceed as in the proof of theorem 1.1 and find a connection A in $\hat{E}$ such that $H_A(\gamma) \neq e$ then we can pull-back this connection to E and obtain a connection in E such that the holonomy of γ is not null. QED

Now let us prove the theorem 1.6. It follows from the following proposition.

Proposition 5.8 *Let γ be a C^N loop such that $H_A(\gamma) = e$ for all connections in the bundle (E, G, M, Π) then there are loops in $E_G(\mathcal{LG})$ arbitrarily C^N close to γ.*

Proof: We first decompose γ in curves forming a web with tassels $T_1, \dots, T_n$, we will do small deformations to these curves to obtain a family of curves that intersects only in a finite number of segments or isolated points. We need to do such deformation in a way that the holonomy of the deformed loop $\hat{\gamma}$ be e for every connection, then the same argument as in proof of theorem 1.1 shows that $\hat{\gamma} \in E_G(\mathcal{LG})$. To accomplish this is sufficient that the group of possible holonomies of the deformed tassel $G_{\hat{T}_i}$ be included in G_{T_i}, and for this is sufficient that the deformation do not take apart curves that intersect.

Fix $\epsilon > 0$, take $\alpha_1, \dots, \alpha_c$ curves in a tassel T in the decomposition of γ and take the parameterization in definition 5.2 such that $\alpha_1(t) = (t, 0, \dots, 0)$, $t \in [0,1]$, we identify points in M with the corresponding points in $\mathbb{R}^d$ in the parameterizations and identify points t in $[0,1]$ with points $(t, 0, \dots, 0)$. Also take a C^∞ function $\rho : \mathbb{R} \to [0,1]$ such that $\rho(x) = 1$ if $x \leq -1$ or $x \geq 1$ and $\rho(x) = 0$ if $-1/2 \leq x \leq 1/2$, let $a_n = \max_{x \in \mathbb{R}} |\rho^{(n)}(x)|$ and let $\rho_{p,\delta}(x) = \rho((x-p)/\delta)$, note that $|\rho^{(n)}_{p,\delta}(x)| \leq a_n/\delta^n$.

We say that a point in the intersection of two curves is singular if it is not in the interior of a common interval of both curves. Let A be the set of singular points of intersection between α_1 and the other curves, because of the way in what was chosen the coordinate system points in A are of the form $(t, 0, \dots, 0)$ and thus we identify them with points of $[0,1]$ which are the values of the parameter of the point taking the parameterization $\alpha_i(t) = (t, f_i(t))$ (where f_i is as in definition 5.2). Let A' be the set of accumulation points of A thus A' is a compact subset of $[0,1]$.

Let $p \in A'$, we define $\mathcal{C}^N(p)$ as the set of curves α_i that have a contact of order N with α_1 in p (that is the first N derivatives of f_i in p are null), note that

if p is accumulation point of intersection points of α_1 and α_i then $\alpha_i \in \mathcal{C}^N(p)$. And define $\mathcal{C}^{<N}(p)$ the set of curves α_i having a contact of order less than N with α_1 in p. We take a $\delta = \delta_p$ such that

1. $\delta < 1$ and if $p \neq t_i$ then $t_i \notin (p-\delta, p+\delta)$ (t_i as in definition 5.2).
2. the curves in $\mathcal{C}^N(p)$ don't intersect the curves in $\mathcal{C}^{<N}$ in other point than p for parameters values in $(p-\delta, p+\delta)$.
3. for each $\alpha_i \in \mathcal{C}^N(p)$, f_i is a C^N function having the first N derivates equal to zero, then there exist $r_p(f_i^{(n)}, \delta)$ such that

$$|f_i^{(n)}(x)| < r_p(f_i^{(n)}, \delta)|x-p|^{N-n}$$

for all $n \leq N$, then take δ such that $r_p(f_i^{(n)}, \delta) < \epsilon$ and

$$a_n r_p(f_i, \delta) + \cdots + C_k^n a_k r_p(f_i^{(n-k)}, \delta) + \cdots + r_p(f_i^{(n)}, \delta) < 2r_p(f_i^{(n)}, \delta).$$

where C_k^n is the binomial coefficient, note that this implies that $|f_i^{(n)}(x)| < \epsilon$ for $x \in (p-\delta, p+\delta)$, $n \leq N$.

Take $P = \{p_1, \ldots, p_m\} \subset A'$ such that $(p_i - \delta_i/2, p_i + \delta_i/2)$ is a finite cover of A'.

For each α_i take $Q_i \subset P$ such that $q \in Q_i$ if and only if $\alpha_i \in \mathcal{C}^N(q)$, for $x \in \mathbb{R}$ let

$$Q_i^-(x) = \{q \in Q_i : q < x \text{ and there is no } q < q' < x \text{ such that } \mathcal{C}^N(q) = \mathcal{C}^N(q')\}$$
$$Q_i^+(x) = \{q \in Q_i : x < q \text{ and there is no } x < q' < q \text{ such that } \mathcal{C}^N(q) = \mathcal{C}^N(q')\}$$

and let $Q_i(x) = Q_i^-(x) \cup Q_i^+(x)$ note that $\#Q_i^{\pm}(x) < 2^c$, next define

$$\rho_i(x) = \prod_{q \in Q_i(x)} \rho_{q,\delta_q}(x)$$

let $\bar{f}_i = f_i \rho_i$ and $\bar{\alpha}_i(t) = (t, \bar{f}_i(t))$. Let $x \in [0, t_i]$, let

$$\{q_1, \ldots, q_k\} = \{q \in Q_i(x) : x \in (q - \delta_q, q + \delta_q)\}$$

and $\delta_j = \delta_{q_j}$ ordered such that $\delta_1 > \delta_2 > \cdots > \delta_k$, let ${}_0f_i = f_i$, ${}_jf_i = {}_{j-1}f_i \rho_{q_j,\delta_j}$ then $\bar{f}_i(x) = {}_kf_i(x)$ we will prove by induction $|{}_jf_i^{(n)}(x)| < 2^j r_{q_k}(f^{(n)}, \delta_j)|x - q_k|^{N-n}$, for $j = 0$ is clear, for $j > 0$

$$\begin{aligned}
|{}_jf_i^{(n)}(x)| &= \left| \sum_{\ell=0}^{n} C_\ell^n \rho_{q_j,\delta_j}^{(\ell)}(x)_{j-1} f_i^{(n-\ell)}(x) \right| \\
&\leq \sum_{\ell=0}^{n} C_\ell^n \left| \frac{a_\ell}{\delta_j^\ell} 2^{j-1} r_{q_k}(f_i^{(n-\ell)}, \delta_j)|x - q_k|^{N-n+\ell} \right| \\
&< 2^j r_{q_k}(f_i^{(n)}, \delta_j)|x - q_k|^{N-n}
\end{aligned}$$

where its used that $|x-q_k|<\delta_k<\delta_j$. Then

$$|\bar{f}_i^{(n)}(x)|<2^k r(f_i^{(n)},\delta_k)|x-q_k|^{N-n}<2^k\epsilon<2^{2^{c+1}}\epsilon.$$

This shows that the distance of α_i and $\bar{\alpha}_i$ is less than $2^{2^{c+1}}\epsilon$ in the C^N topology, and this construction removed all the accumulation points of singular intersection points between α_1 and $\bar{\alpha}_2$, then α_1 and $\bar{\alpha}_2$ intersect in a finite number of intervals or isolated points. We have to see that $G_{\bar{T}}$, the set of possible values for holonomies of the curves $\bar{\alpha}_i$, is not larger than G_T, to show this it is sufficient to show that if $\alpha_i(t)=\alpha_j(t)$ then $\bar{\alpha}_i(t)=\bar{\alpha}_j(t)$.

Let $Q_i'(x)=\{q\in Q_i(x): x\in(q-\delta_q,q+\delta_q)\}$, then

$$\rho_i(x)=\prod_{q\in Q_i'(x)}\rho_{q,\delta_q}(x)$$

if $q\in Q_i(x)$ and $q\notin Q_j(x)$ then $\alpha_i\in\mathcal{C}^N(q)$ and $\alpha_j\in\mathcal{C}^{<N}(q)$ thus α_i do not intersect α_j in $(q-\delta_q,q+\delta_q)$ and then $\alpha_i(x)\neq\alpha_j(x)$. This shows that if $\alpha_i(x)=\alpha_j(x)$ then $Q_i'(x)=Q_j'(x)$ then $\rho_i(x)=\rho_j(x)$ and this implies $\bar{\alpha}_i(x)=\bar{\alpha}_j(x)$. QED

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